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Spin, angular momentum and spin-statistics for a relativistic quantum many body system*

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Abstract The adaptation of Wigner's induced representation for a relativistic quantum theory making possible the construction of wavepackets and admitting covariant expectation values for the coordinate operator x^μ introduces a foliation on the Hilbert space of states. The spin-statistics relation for fermions and bosons implies the universality of the parametrization of orbits of the induced representation, implying that all particles within identical particle sets transform under the same $SU(2)$ subgroup of the Lorentz group, and therefore their spins and angular momentum states can be computed using the usual Clebsch-Gordan coefficients associated with angular momentum. Important consequences, such as entanglement for subsystems at unequal times, covariant statistical correlations in many body systems, and the construction of relativistic boson and fermion statistical ensembles, as well as implications for the foliation of the Fock space and for quantum field theory are briefly discussed.

1. Introduction

The spin of a particle in a nonrelativistic framework corresponds to the lowest dimensional nontrivial representation of the rotation group; the generators are the Pauli matrices σ_i divided by two, the generators of the fundamental representation of the double covering of $SO(3)$. The self-adjoint operators that are the generators of this group measure intrinsic angular momentum and are associated with magnetic moments.

In the nonrelativistic quantum theory, the spin states of a two or more particle system are defined by combining the spins of these particles at equal time using appropriate Clebsch-Gordan coefficients [1] at each value of the time. The restriction to equal time follows from the tensor product form of the representation of the quantum states for a many body problem [2]. This correlation at equal time in the nonrelativistic quantum theory is the source of the famous Einstein-Podolsky-Rosen discussion[3] and provides an important model for quantum information transfer.

The standard Pauli description of a particle with spin is not, however, relativistically covariant, but Wigner [4] has shown how to describe this dynamical property of a particle

* This paper is dedicated to the memory of Constantin Piron.

in a covariant way. The method developed by Wigner has provided the foundation for what is now known as the theory of induced representations [5], with very wide applications, including a very powerful approach to finding the representations of noncompact groups.

The formulation of Wigner[4] is, however, not appropriate for application to quantum theory, since it does not preserve, as we shall explain below, the covariance of the expectation value of coordinate operators. We first briefly review Wigner's method in its original form, and show how the difficulties arise. We then review the extension of Wigner's approach necessary to describe the spin of a particle in the framework of the manifestly covariant theory of Stueckelberg, Horwitz and Piron [6](SHP). We then show that the observed correlation of spin and statistics for identical particles necessitates a structure for which the Hilbert space of states of a many body system of identical particles is represented as a direct integral over all values of a (normalized) timelike vector, a structure called *foliation*. The relativistic many body system then admits the description of total spin (in general, total angular momentum) states through the computation of Clebsch-Gordan coefficients as in the nonrelativistic case, and implies correlations between the spins of the particles much in the same way. It has been shown that relativistically covariant canonical ensembles can be constructed in the framework of the SHP theory[7] as well as a corresponding Boltzmann transport theory[8]. The results that we achieve here admit an extension of these results to particles with spin; the results obtained in these earlier works may be embedded in the foliation implied by the accommodation of spin. The full development of the consequences for thermodynamics and phase transitions will be left for succeeding studies.

The foliation universally induced in the representation for physical many body systems applies both to fermion and boson sectors of the full Fock space, and therefore to the quantum fields. Further development of the consequences of this structure will also be left for succeeding publications,

As Wigner [4] has shown (see also the detailed discussion in Weinberg [9]), constructing a representation of the Lorentz group by inducing a representation on the stability group of the (timelike) four-momentum, one obtains a representation $\psi(p, \sigma)$ with the transformation property

$$\psi'(p, \sigma) = \psi(\Lambda^{-1}p, \sigma') D_{\sigma', \sigma}(\Lambda, p), \quad (1)$$

under the action of the Lorentz group, taking into account the spin degrees of freedom of the wavefunction, where the matrix transformation factor (Wigner's "little group"[4]) is constructed of the 2×2 matrices of $SL(2, C)$.

The presence of the p -dependent matrices representing the spin of a relativistic particle in the transformation law of the wave function, however, destroys the covariance, in a relativistic quantum theory, of the expectation value of the coordinate operators [11] in states transforming as in (1). To see this, consider the expectation value of the dynamical variable x^μ [6], *i.e.*

$$\langle x^\mu \rangle = \Sigma_\sigma \int d^4p \psi(p, \sigma)^\dagger i \frac{\partial}{\partial p_\mu} \psi(p, \sigma). \quad (2)$$

A Lorentz transformation would introduce the p -dependent 2×2 unitary transformation on the function $\psi(p, \sigma)$, and the derivative with respect to momentum would destroy the covariance property that we would wish to see of the expectation value $\langle x^\mu \rangle$.

It is also not possible, in this framework, to form wave packets of definite spin by ($4D$) Fourier transform over the momentum variable, since this would add functions over different parts of the orbit, with a different $SU(2)$ at each point.

These problems were solved [11] by inducing a representation of the spin on a timelike unit vector, say, n^μ in place of the four-momentum.

Using a representation induced on a timelike vector n^μ , which is independent of x^μ or p^μ ** permits the linear superposition of momentum eigenstates to form wave packets of definite spin, and admits the construction of definite spin states for many body relativistic systems. In the following, we show how such a representation can be constructed, and discuss some of its dynamical implications.

2. Induced representation on timelike vector n^μ

We briefly review here the construction given in [11] in order to make clear the nature of the resulting foliation of the Hilbert space. Let us define,

$$|n, \sigma, x \rangle \equiv U(L(n))|n_0, \sigma, x \rangle, \quad (3)$$

where we may admit a dependence on x (or, through Fourier transform, on p). Here, we distinguish the action of $U(L(n))$ from the general Lorentz transformation $U(\Lambda)$; $U(L(n))$ acts only on the manifold of $\{n^\mu\}$. Its infinitesimal generators are given by

$$M_n^{\mu\nu} = -i(n^\mu \frac{\partial}{\partial n_\nu} - n^\nu \frac{\partial}{\partial n_\mu}), \quad (4)$$

while the generators of the transformations $U(\Lambda)$ act on the full space of both the n^μ and the x^μ (as well as p^μ); its generators are given by

$$M^{\mu\nu} = M_n^{\mu\nu} + (x^\mu p^\nu - x^\nu p^\mu). \quad (5)$$

The two terms of the generator commute, and therefore the full group is a (diagonal) direct product.

We now investigate the properties of a total Lorentz transformation, *i.e.*, as in Wigner's procedure[4],

$$U(\Lambda^{-1})|n, \sigma, x \rangle = U(L(\Lambda^{-1}n))(U^{-1}(L(\Lambda^{-1}n))U(\Lambda^{-1}))|n_0, \sigma, x \rangle, \quad (6)$$

Now, consider the conjugate of (6),

$$\langle n, \sigma, x | U(\Lambda) = \langle n_0, \sigma, x | (U(L^{-1}(n))U(\Lambda)U(L(\Lambda^{-1}n)))U^{-1}(L(\Lambda^{-1}n)). \quad (7)$$

The operator in the first factor (in parentheses) preserves n_0 , and therefore corresponds to an element of the little group associated with n^μ which may be represented by

** Note that the resulting Stueckelberg type wave functions $\psi_n(x, \sigma)$ are local [6] and do not have the non-local properties discussed by Newton and Wigner [10].

the matrices of $SL(2, C)$. It also, due to the factor $U(\Lambda)$ (for which the generators are those of the Lorentz group acting both on n and x (or p)), takes $x \rightarrow \Lambda^{-1}x$ in the conjugate ket on the left. Taking the product on both sides with $|\psi\rangle$, we obtain

$$\langle n, \sigma, x | U(\Lambda) | \psi \rangle = \langle \Lambda^{-1}n, \sigma', \Lambda^{-1}x | \psi \rangle D_{\sigma', \sigma}(\Lambda, n), \quad (8)$$

or [11]

$$\psi'_{n, \sigma}(x) = \psi_{\Lambda^{-1}n, \sigma'}(\Lambda^{-1}x) D_{\sigma', \sigma}(\Lambda, n). \quad (9)$$

where

$$D(\Lambda, n) = L^{-1}(n) \Lambda L(\Lambda^{-1}n), \quad (10)$$

with Λ and $L(n)$ the corresponding 2×2 matrices of $SL(2, C)$.

It is clear that, with this transformation law, one may take the Fourier transform to obtain the wave function in momentum space, and conversely. The matrix $D(\Lambda, n)$ is an element of $SU(2)$, and therefore linear superpositions over momenta or coordinates maintain the definition of the particle spin for each n^μ , and interference phenomena for relativistic particles with spin may be studied consistently. Furthermore, if two or more particles with spin are represented in representations induced on n^μ , at the same value of n^μ on their respective orbits, and therefore in the *same* $SU(2)$ representation, their spins can be added by the standard methods with the use of Clebsch-Gordan coefficients. This method therefore admits the treatment of a many body relativistic system with spin, as in the proposed experiment of Palacios, *et al* [12].

Our assertion of the unitarity of the n -dependent part of the transformation has assumed that the integral measure on the Hilbert space, to admit integration by parts, is of the form $d^4n d^4x$, where the support of the wave functions on n^μ is in the timelike sector. The action of the generator of Lorentz transformations on n^μ maintains the normalization of $n^\mu n_\mu$, which we shall take to be -1 in our discussion of the Dirac representation for the wave function. Although the timelike vector n^μ , in many applications, is degenerate, it carries a probability interpretation under the norm, and may play a dynamical role (for example, as for the spacelike inducing vector for the two body bound state problem in the covariant Zeeman formulation of ref.[13]).

There are two fundamental representations of $SL(2, C)$ which are inequivalent[14]. Multiplication of a two dimensional spinor representing one of these by the operator $\sigma \cdot p$, expected to occur in any dynamical theory, results in an object transforming like the other representation, and therefore the state of lowest dimension in spinor indices of a physical system should contain both representations. As we shall emphasize, however, in our treatment of the more than one particle system, for the rotation subgroup, both of the fundamental representations yield the same $SU(2)$ matrices up to a unitary transformation, and therefore the Clebsch-Gordan decomposition of the product state into irreducible representations may be carried out independently of which fundamental $SL(2, C)$ representation is associated with each of the particles. This is therefore true for the Dirac representation, incorporating both fundamental representations, as we see in Eq. (11).

As in ref. [11], one finds the Dirac spinor

$$\psi_n(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} L(n) \hat{\psi}_n(x) \\ \underline{L}(n) \hat{\phi}_n(x) \end{pmatrix}, \quad (11)$$

which transforms as

$$\psi'_n(x) = S(\Lambda)\psi_{\Lambda^{-1}n}(\Lambda^{-1}x) \quad (12)$$

where $S(\Lambda)$ is a (nonunitary) transformation generated infinitesimally, as in the standard Dirac theory (see, for example, Bjorken and Drell[16]), by $\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

Following the arguments of [11], one can construct, in the presence of a $U(1)$ gauge field, the covariant Hamiltonian

$$K = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M}\Sigma_n^{\mu\nu}F_{\mu\nu}(x) - eA_5, \quad (13)$$

where

$$\Sigma_n^{\mu\nu} = \Sigma^{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu, \quad (14)$$

and $K^\mu = \Sigma^{\mu\nu}n_\nu$. The A_5 field arises as a compensation field for the τ derivative in the Stueckelberg-Schrödinger equation [17]. In general, in this framework, the A^μ and A^5 fields may depend on τ , since they correspond to gauge compensation fields for the local gauge transformation $\psi_\tau(x) \rightarrow \exp i\Lambda(x, \tau)\psi_\tau(x)$. The τ -independent Maxwell fields correspond to the zero mode of the A^μ fields used here [17]. The currents constructed from the Lagrangian associated with (13) are, according to (11), also foliated, and therefore the fields A_μ, A_5 generated by these currents, will be foliated as well. We shall discuss this structure further in a succeeding article.

The expression (13) is quite similar to that of the second order Dirac operator; it is, however, Hermitian and has no direct electric coupling to the electromagnetic field in the special frame for which $n^\mu = (1, 0, 0, 0)$ in the minimal coupling model we have given here (note that in his calculation of the anomalous magnetic moment[18], Schwinger puts the electric field to zero; a non-zero electric field would lead to a non-Hermitian term in the standard Dirac propagator, the inverse of the Klein-Gordon square of the interacting Dirac equation). Note that in the derivation of the anomalous magnetic moment given by Bennett[19], this restriction is not necessary since the generator of the interacting motion is intrinsically Hermitian.

The matrices $\Sigma_n^{\mu\nu}$ are, in fact, a relativistically covariant form of the Pauli matrices.

To see this[11], we note that the quantities K^μ and $\Sigma_n^{\mu\nu}$ satisfy the commutation relations

$$\begin{aligned} [K^\mu, K^\nu] &= -i\Sigma_n^{\mu\nu} \\ [\Sigma_n^{\mu\nu}, K^\lambda] &= -i[(g^{\nu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\mu\lambda} + n^\mu n^\lambda)K^\nu, \\ [\Sigma_n^{\mu\nu}, \Sigma_n^{\lambda\sigma}] &= -i[(g^{\nu\lambda} + n^\nu n^\lambda)\Sigma_n^{\mu\sigma} - (g^{\sigma\mu} + n^\sigma n^\mu)\Sigma_n^{\lambda\nu} \\ &\quad - (g^{\mu\lambda} + n^\mu n^\lambda)\Sigma_n^{\nu\sigma} + (g^{\sigma\nu} + n^\sigma n^\nu)\Sigma_n^{\lambda\mu}]. \end{aligned} \quad (15)$$

Since $K^\mu n_\mu = n_\mu \Sigma^{\mu\nu} = 0$, there are only three independent K^μ and three $\Sigma_n^{\mu\nu}$. The matrices $\Sigma_n^{\mu\nu}$ are a covariant form of the Pauli matrices, and the last of (15) is the Lie algebra of $SU(2)$ in the spacelike surface orthogonal to n^μ . The three independent K^μ correspond to the non-compact part of the algebra which, along with the $\Sigma_n^{\mu\nu}$ provide a representation of the Lie algebra of the full Lorentz group. The covariance of this representation follows from

$$S^{-1}(\Lambda)\Sigma_{\Lambda n}^{\mu\nu}S(\Lambda)\Lambda_\mu^\lambda\Lambda_\nu^\sigma = \Sigma_n^{\lambda\sigma}. \quad (16)$$

In the special frame for which $n^\mu = (1, 0, 0, 0)$, Σ_n^{ij} become the Pauli matrices $\frac{1}{2}\sigma^k$ with (i, j, k) cyclic, and $\Sigma_n^{0j} = 0$. In this frame there is no direct electric interaction with the spin in the minimal coupling model (14). We remark that there is, however, a natural spin coupling which becomes pure electric in the special frame[11], given by (in gauge covariant form)

$$i[K_T, K_L] = -ie\gamma^5(K^\mu n^\nu - K^\nu n^\mu)F_{\mu\nu}. \quad (17)$$

Note that the matrices

$$\gamma_n^\mu = \gamma_\lambda \pi^{\lambda\mu}, \quad (18)$$

with the projection

$$\pi^{\lambda\mu} = g^{\lambda\mu} + n^\lambda n^\mu, \quad (19)$$

appearing in (15), plays an important role in the description of the dynamics in the induced representation. In (13), the existence of projections on each index in the spin coupling term implies that $F^{\mu\nu}$ can be replaced by $F_n^{\mu\nu}$, a tensor projected into the foliation subspace. As we shall see, this foliation, induced by the spin, has a profound effect on the tensor products (and therefore on the full Fock space) of identical particle systems, both in the boson and fermion sectors§.

3. The many body problem with spin, and spin-statistics

As in the nonrelativistic quantum theory, one represents the state of an N -body system in terms of a basis given by the tensor product of N one-particle states, each an element of a one-particle Hilbert space. The general state of such an N -body system is given by a linear superposition over this basis [21]. Second quantization then corresponds to the construction of a Fock space, for which the set of all N body states, for all N , are imbedded in a large Hilbert space for which operators that change the number N are defined [2]. We shall discuss this structure in this section, and show, with our discussion of the relativistic spin given in the previous section, that the spin of a relativistic many-body system can be well-defined and, furthermore, that the quantum fields associated with the particles of the system carry the induced foliation structure.

In order to construct the tensor product space corresponding to the many-body system, we consider, as for the nonrelativistic theory, the product of wave functions which are elements of isomorphic Hilbert spaces. In the nonrelativistic theory, this corresponds to functions at equal time; in the relativistic theory, the functions are at equal τ . Thus, in the relativistic theory, there are correlations at unequal t , within the support of the Stueckelberg wave functions. Moreover, for particles with spin we argue, as a consequence of the spin-statistics relation, that in the induced representation, these functions must be taken at *identical values of n^μ* , i.e., taken at the same point on the orbits of the induced representation of each particle:

§ Note that for the $SO(1, 1)$ covariant generalization of one dimensional systems treated, for example, by methods of the Bethe ansatz [20], the relation between spin and statistics is not so direct, and therefore this problem requires a separate discussion.

Statement: Identical particles must be represented in tensor product states by wave functions not only at equal τ but also at equal n^μ .

The proof of this statement lies in the observation that the spin-statistics relation appears to be a universal fact of nature. The elementary proof of this statement, for example, for a system of two spin 1/2 particles, is that a π rotation of the system introduces a phase factor of $e^{i\frac{\pi}{2}}$ for each particle, thus introducing a minus sign for the two body state. However, the π rotation is equivalent to an interchange of the two identical particles. This argument rests on the fact that each particle is in the same representation of $SU(2)$, which can only be achieved in the induced representation with the particles at the same point on their respective orbits. We therefore see that identical particles must carry the same value of n^μ , and the construction of the N -body system must follow this rule† It therefore follows that the two body relativistic system can carry a spin computed by use of the usual Clebsch-Gordon coefficients, and entanglement would follow even at unequal time (within the support of the equal τ wave functions), as in the proposed experiment of Palacios *et al*[12]. This argument can be followed, as we shall do in Section 4, for arbitrary N , and therefore the Fock space of the quantum field theory carries the properties usually associated with fermion (or boson) fields. with the entire Fock space foliated over the orbit of the inducing vector n^μ . .

Although, due to the Newton-Wigner problem [10] noted above, the solutions of the Dirac equation are not suitable for the covariant local description of a quantum theory, the functions constructed in (11) can form the basis of a consistent, local (off-shell) covariant quantum theory.

To show how the many body Fock space develops, we start by constructing a two body Hilbert space in the framework of the relativistic quantum theory. The states of this two body space are given by linear combinations over the product wave functions, where the wave functions are given by Dirac functions of the type described in (11), *i.e.*, temporarily suppressing the indices n, τ ,

$$\psi_{ij}(x_1, x_2) = \psi_i(x_1) \times \psi_j(x_2), \quad (20)$$

where $\psi_i(x_1)$ and $\psi_j(x_2)$ are elements of the one-particle Hilbert space \mathcal{H} . Let us introduce the notation, often used in differential geometry, that

$$\psi_{ij}(x_1, x_2) = \psi_i \otimes \psi_j(x_1, x_2), \quad (21)$$

identifying the arguments according to a standard ordering. Then, without specifying the spacetime coordinates, we can write

$$\psi_{ij} = \psi_i \otimes \psi_j, \quad (22)$$

† Note that symmetrization and antisymmetrization can, of course, be carried out with factors in the tensor product on any sequence in n , but the symmetry properties would not then correspond to the phases associated with spin.

formally, an element of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The scalar product is carried out by pairing the elements in the two factors according to their order, since it corresponds to integrals over x_1, x_2 , *i.e.*,

$$(\psi_{ij}, \psi_{k,\ell}) = (\psi_i, \psi_k)(\psi_j, \psi_\ell). \quad (23)$$

For two identical particle states satisfying Bose-Einstein or Fermi-Dirac statistics, we must write, according to our argument given above,

$$\psi_{ijn} = \frac{1}{\sqrt{2}}[\psi_{in} \otimes \psi_{jn} \pm \psi_{jn} \otimes \psi_{in}]. \quad (24)$$

This expression has the required symmetry or antisymmetry only if both functions are on the same points of their respective orbits in the induced representation. Furthermore, they transform under the *same* $SU(2)$ representation of the rotation subgroup of the Lorentz group, and thus for spin 1/2 particles, under a π spatial rotation (defined by the space orthogonal to the timelike vector n^μ) they both develop a phase factor $e^{i\frac{\pi}{2}}$. The product results in an overall negative sign. As in the usual quantum theory, this rotation corresponds to an interchange of the two particles, but here with respect to a “spatial” rotation around the vector n^μ . The spacetime coordinates in the functions are rotated in this (foliated) subspace of spacetime, and correspond to an actual exchange of the positions of the particles in space time, as in the formulation of the standard spin-statistics theorem. It therefore follows that the interchange of the particles occurs in the foliated space defined by n^μ . For identical bosonic particles, the π rotation produces a positive sign. These conclusions are valid for unequal times that lie in support of the SHP wave functions (at equal τ). We therefore have the

Statement: The antisymmetry of identical half-integer spin (fermionic) particles remains at unequal times (within the support of the wave functions). This is true for the symmetry of identical integer spin (bosonic) particles as well.

Furthermore, the construction we have given enables us to define the spin of a many body system, even if the particles are relativistic and moving arbitrarily with respect to each other. Since all particles with representations on a common n^μ of their orbits transform in the spacelike submanifold orthogonal to n^μ under the same $SU(2)$, it is also true that

*Statement: The spin of an N -body system of identical particles is well-defined, independent of the state of motion of the particles of the system, by the usual laws of combining representations of $SU(2)$, *i.e.*, with the usual Clebsch-Gordan coefficients, since the states of all the particles in the system are in induced representations at the same point of the orbit n^μ .*

Thus, for example, in the quark model for hadrons, the total spin of the hadron can be computed from the spins (and orbital angular momenta projected into the foliated space) of the individual quarks using the usual Clebsch-Gordan coefficients even if they are in significant relative motion, within the same $SU(2)$; a similar conclusion would be

valid for nucleons in a nucleus even at high excitation. The validity of spin assignments in high energy scattering would provide an important example of such quantum mechanical correlations.

In the course of our construction, we have seen that the foliation of the spacetime follows from the arguments based in the representations of a relativistic particle with half-integer spin. However, as we have remarked, our considerations of the nature of identical particles, and their association with the spin statistics properties observed in nature, require that the foliation persists in the bosonic sector as well, where a π rotation, exchanging two particles, must be in a definite representation of the rotation group, specified by the foliation vector n^μ , to achieve a positive sign. Since there is no extra phase (corresponding to integer representations of the $SU(2)$) for the Bose-Einstein case, the boson symmetry can then be extended to a covariant symmetry with important implications, for example, for the statistical mechanics of relativistic boson systems, as, for example, Bose-Einstein condensation.

We remark in this connection that the Cooper pairing[22] of superconductivity must be between electrons on the same point of their induced representation orbits, so that the superconducting state is defined on the corresponding foliation of spacetime as well. The resulting (quasi-) bosons have the identical particle properties inferred from our discussion of the boson sector. As remarked above, the two electrons of the Cooper pair may not be at equal time, a result which may be accessible to experiment. A similar remark applies to the Josephson effect[23] (where a single gate may be opened at two successive times, as in the Lindner experiment[24]).

These results have, moreover, important implications in atomic and molecular physics, for example, for the construction of the exchange interaction.

4. Quantum Fields

We now extend our argument for the finite Fock space to the general structure of quantum field theory.

The N body state of Fermi-Dirac particles can be written as (the N body boson system should be treated separately since the normalization conditions are different, but we give the general result below)

$$\Psi_{nN} = \frac{1}{N!} \Sigma(-)^P P \psi_{nN} \otimes \psi_{nN-1} \otimes \cdots \psi_{n1}, \quad (25)$$

where the permutations P are taken over all possibilities, and no two functions are equal. By the arguments given above, any pair of particle wave functions in this set have the Fermi-Dirac symmetry properties. We may now think of such a function as an element of a larger Hilbert space, the *Fock space*, which contains all values of the number N . On this space, one can define an operator that adds another particle (in the tensor product), performs the necessary antisymmetrization, and changes the normalization appropriately. This operator is called a *creation operator*, which we shall denote by $a^\dagger(\psi_{nN+1})$ and has the property that

$$a^\dagger(\psi_{nN+1}) \Psi_{nN} = \Psi_{nN+1}, \quad (26)$$

now to be evaluated on the manifold $(x_{N+1}, x_N, x_{N-1} \dots x_1)$. Taking the scalar product with some $N + 1$ particle state Φ_{nN+1} in the Fock space, we see that

$$(\Phi_{nN+1}, a^\dagger(\psi_{nN+1})\Psi_{nN}) \equiv (a(\psi_{nN+1})\Phi_{nN+1}, \Psi_{nN}), \quad (27)$$

thus defining the *annihilation* operator $a(\psi_{nN+1})$.

The existence of such an annihilation operator, as in the usual construction of the Fock space, *e.g.*, [2], implies the existence of an additional element in the Fock space, the *vacuum*, or the state of no particles. The vacuum defined in this way lies in the foliation labelled by n^μ . The covariance of the construction, however, implies that, since all sectors labelled by n^μ are connected by the action of the Lorentz group, that this vacuum is a vacuum for any n^μ , *i.e.*, the vacuum $\{\Psi_{n0}\}$ over all n^μ is Lorentz invariant.

The commutation relations of the annihilation-creation operators can be easily deduced from a low dimensional example, following the method used in the nonrelativistic quantum theory[2]. Consider the two body state (24) (we use the antisymmetric form here), and apply the creation operator $a^\dagger(\psi_{n3})$ to create the three body state

$$\begin{aligned} \Psi(\psi_{n3}, \psi_{n2}, \psi_{n1}) = \frac{1}{\sqrt{3!}} \{ & \psi_{n3} \otimes \psi_{n2} \otimes \psi_{n1} + \psi_{n1} \otimes \psi_{n3} \otimes \psi_{n2} \\ & + \psi_{n2} \otimes \psi_{n1} \otimes \psi_{n3} - \psi_{n2} \otimes \psi_{n3} \otimes \psi_{n1} \\ & - \psi_{n1} \otimes \psi_{n2} \otimes \psi_{n3} - \psi_{n3} \otimes \psi_{n1} \otimes \psi_{n2} \} \end{aligned} \quad (28)$$

One then takes the scalar product with the three body state

$$\begin{aligned} \Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) = \frac{1}{\sqrt{3!}} \{ & \phi_{n3} \otimes \phi_{n2} \otimes \phi_{n1} + \phi_{n1} \otimes \phi_{n3} \otimes \phi_{n2} \\ & + \phi_{n2} \otimes \phi_{n1} \otimes \phi_{n3} - \phi_{n2} \otimes \phi_{n3} \otimes \phi_{n1} \\ & - \phi_{n1} \otimes \phi_{n2} \otimes \phi_{n3} - \phi_{n3} \otimes \phi_{n1} \otimes \phi_{n2} \} \end{aligned} \quad (29)$$

Carrying out the scalar product term by term, and picking out the terms corresponding to the scalar product of some function with the two body state

$$\frac{1}{\sqrt{2}} \{ \psi_{n2} \otimes \psi_{n1} - \psi_{n1} \otimes \psi_{n2} \} \quad (30)$$

one finds that the action of the operator $a(\psi_{n3})$ on the state $\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1})$ is given by

$$\begin{aligned} a(\psi_{n3})\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) = & (\psi_{n3}, \phi_{n3})\phi_{n2} \otimes \phi_{n1} \\ & - (\psi_{n3}, \phi_{n2})\phi_{n3} \otimes \phi_{n1} + (\psi_{n3}, \phi_{n1})\phi_{n3} \otimes \phi_{n2}, \end{aligned} \quad (31)$$

i.e., the annihilation operator acts like a derivation with alternating signs due to its fermionic nature; the relation of the two and three body states we have analyzed has a direct extension to the N -body case. The action of boson annihilation-creation operators can be derived in a similar way.

Applying these operators to N and $N + 1$ particle states, one find directly their commutation and anticommutation relations

$$[a(\psi_n), a^\dagger(\phi_n)]_\mp = (\psi_n, \phi_n), \quad (32)$$

where the \mp sign, corresponds to commutator or anticommutator for the boson or fermion operators. If the functions ψ_n, ϕ_n belong to a normalized orthogonal set $\{\phi_{nj}\}$, then

$$[a(\phi_{ni}), a^\dagger(\phi_{nj})]_\mp = \delta_{ij}, \quad (33)$$

Let us now suppose that the functions ϕ_{nj} are plane waves in spacetime, *i.e.*, in terms of functions

$$\phi_{np}(x) = \frac{1}{(2\pi)^2} e^{-ip^\mu x_\mu}, \quad (34)$$

so that

$$(\phi_{np}, \phi_{np'}) = \delta^4(p - p'). \quad (35)$$

The quantum fields are then constructed as follows. Define

$$\phi_n(x) \equiv \int d^4p a(\phi_{np}) e^{ip^\mu x_\mu}. \quad (36)$$

It then follows that, by the commutation (anticommutation) relations (32), these operators obey the relations

$$[\phi_n(x), \phi_n^\dagger(x')]_\mp = \delta^4(x - x'), \quad (37)$$

corresponding to the commutation relations of bose and fermion *fields* (we suppress the spinor indices here, arising from the spinor form which must be used for (34)). Under Fourier transform, one finds the commutation relations in momentum space

$$[\phi_n(p), \phi_n^\dagger(p')]_\mp = \delta^4(p - p') \quad (38)$$

The relation of these quantized fields with those of the usual on-shell quantum field theories can be understood as follows. Let us suppose that the fourth component of the energy-momentum is $E = \sqrt{\mathbf{p}^2 + m^2}$, where m^2 is close to a given number, the on-shell mass of a particle. Then, noting that $dE = \frac{dm^2}{2E}$, if we multiply both sides of (38) by dE and integrate over the small neighborhood of m^2 occurring in both E and E' , the delta function $\delta(E - E')$ on the right hand side integrates to unity. On the left hand side, there is a factor of $dm^2/2E$, and we may absorb $\sqrt{dm^2}$ in each of the field variables, obtaining (with $\varphi_n(\mathbf{p}) \equiv \sqrt{dm^2} \phi_n(p)$ on shell)

$$[\varphi_n(\mathbf{p}), \varphi_n^\dagger(\mathbf{p}')]_\mp = 2E \delta^3(\mathbf{p} - \mathbf{p}'), \quad (39)$$

the usual formula for on-shell quantum fields.

We remark that these algebraic results have been constructed in the foliation involved in the formulation of a consistent theory of relativistic spin, therefore admitting the action

of the $SU(2)$ group (in the Dirac representation (12), $S(\Lambda)$ has the form $e^{i\Sigma_n^{\mu\nu}\omega_{\mu\nu}}$ for $\omega_{\mu\nu}$ parameters corresponding to the $SU(2)$ subgroup leaving n^μ invariant) for a many body system, applicable for unequal times, within the support of the Stueckelberg wave functions at equal τ .

We have discussed here the construction of quantum fields as they emerge from the structure of a Fock space. Local observables can be formed from the Hermitian operators built with these fields. According to the methods generally attributed to Schwinger[25] and Tomonaga[26] (see the book of Jauch and Rohrlich[27], for example, for a discussion of the ideas and additional references), a quantum state is defined by assigning values to the spectra of a complete set of such local observables which necessarily commute, according to the causal nature of measurements, if they are associated with a spacelike surface. The sequence of spacelike surfaces then forms a parametrization of the evolution of such states §§(the basis of the Schwinger-Tomonaga equation); it follows from our considerations that, for states of identical particles, the set of local observables are defined on the foliation provided by the inducing parameter n^μ , and therefore the Schwinger-Tomonaga state lies on this foliation as well. Furthermore, since the local fields, in Heisenberg picture, evolve unitarily in τ , and the corresponding spacelike surfaces are isomorphic, a correspondence can be established between τ and an invariant parameter labelling the sequence of spacelike surfaces. Moreover, it is clear from (15) that the action of the operators $\Sigma_n^{\mu\nu}$, due to the occurrence of the projections $g^{\nu\lambda} + n^\nu n^\lambda$ in the coefficients of the Lie algebra, correspond to rotations in the spacelike surface orthogonal to the timelike vector n^μ (as we have remarked, in the frame for which $n^\mu = (1, 0, 0, 0)$, these operators reduce to the ordinary Pauli matrices). Together with the operators K^μ , they constitute a representation of the Lorentz group, forming the fundamental representation of a group oriented with its maximal compact subgroup, corresponding to the $SU(2)$ little group of Wigner, acting on the wavefunctions, and the corresponding quantum fields, as a rotation in the spacelike surface orthogonal to n^μ . We may therefore identify the spacelike surfaces on which the quantum fields are defined with the spacelike surfaces on which the little group induces rotations (as in the nonrelativistic theory). Local variations in the spacelike surfaces, contemplated by Schwinger and Tomonaga, then correspond as well to local variations in the orbit of the induced representation, clearly preserving the local commutation and anticommutation relations††. This structure will be examined in more detail elsewhere.

5. Discussion and Conclusions

We have applied the method of induced representations of Wigner (1939)[4] for the description of a relativistic particle with spin; adapted to a structure useful in the framework of a relativistic quantum theory. The method requires that the representation be induced on an arbitrary timelike vector instead of the four-momentum used by Wigner. Each point on the orbit of this timelike vector is associated with an $SU(2)$, its stability subgroup. Two particles at the same point on their respective orbits then transform, under rotations in

§§ In a variational sense.

†† I am grateful to one of the referees for raising the question of quantization on a set of spacelike surfaces

the space orthogonal to the timelike vector with the same $SU(2)$ and therefore their spins can be added with the usual Clebsch-Gordan coefficients. The existence of the relation between spin and statistics in nature implies that the fermionic antisymmetry between any pair of identical particles, associated with a π rotation of the two-body subsystem can be valid only for particles on the same points of their respective orbits. This result introduces a foliation of the whole Fock space constructed from the many-body tensor product, and therefore of the corresponding quantum field theory for both bosons and fermions; we discuss the correspondence of this foliation with the structure of quantum field theory defined on a sequence of spacelike surfaces[25][26][27]. We shall discuss the consequences for fermion and boson fields, as well as the (foliated) radiation fields generated by their currents, more fully in a succeeding publication.

One can, in this way, compute the total spin state of a many-body system, provided all particles are at the same point on their respective orbits of the timelike inducing vector, as required for identical particle systems. Furthermore, as in the proposed experiment of Palacios *et al*[12], the spin entanglements induced in this way would exist for particles at equal world time τ , but not restricted, as in standard nonrelativistic mechanics, to equal time; these correlations should be seen, according to this theory, for particles at non-equal times within the support of the Stueckelberg wave functions. The analysis of Palacios *et al*[12] assumes coherence in time and uses, as in the analysis of Lindner *et al*[24] of their experiment, time dependent solutions of the nonrelativistic Schrödinger equation. This treatment is not consistent with the basic foundations of the quantum theory[28], but is expected to provide, as in the Lindner *et al*[24] experiment, a good approximation under some circumstances[28].

The correlations implied by the existence of Cooper pairs, forming the foundation of the theory of superconductivity, existing, according to the nonrelativistic quantum theory only at equal times, are predicted by the SHC theory, as a consequence of the work reported here, to be maintained at unequal times. The theory can therefore be generalized to be consistent with relativistic covariance. In a similar way, we predict that the interference phenomena associated with the Josephson effect[23] would be maintained if the two gates are open at different times, or with a single gate opened at two times, with a result similar to that of the Lindner *et al* experiment[24]. Such a result would be a remarkable generalization of the Josephson effect.

We finally remark that the Boltzmann counting leading to the relativistic Bose-Einstein and Fermi-Dirac distributions carried out in [7] corresponds, from the point of view presented here, to the use of foliated wave functions; for the free boson gas or fermion gas, there would be no essential difference in the results, but the distribution functions would appear with nontrivial foliation. Furthermore, the Wigner functions in terms of which the quantum transport was computed in ref [8] would be foliated for the treatment of identical particle systems, as would the density matrix and the corresponding BBGKY hierarchy and the resulting Fokker-Planck equations. The consequences of this foliation for quantum statistical mechanics will be discussed more completely, as well, in succeeding publications.

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References

1. See, for example, L.C. Biedenharn and J.D. Louck, *The Racah-Wigner Algebra in Quantum Theory* Addison-Wesley, Reading Massachusetts (1981)
2. G. Baym, *Lectures on Quantum Mechanics*, W.A. Benjamin, New York (1969).
3. A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **41**, 1881 (1978).
4. E. Wigner, Ann. Math. **40**, 149 (1939).
5. G.W. Mackey, *Induced Representations of Groups and Quantum Mechanics*, W.A. Benjamin, New York (1968).
6. E.C.G. Stueckelberg, Helv. Phys. Acta **14**, 322-323, 588-594 (1941); **15**, 23-27 (1942).
L.P. Horwitz and C. Piron, Helv. Phys. Acta **46**, 316-326 (1973).
7. L.P. Horwitz, W.C. Schieve and Piron, Ann. Phys. **137**, 306-340 (1981).
8. L.P. Horwitz, S. Shashoua and W.C. Schieve, Physica A **161**, 300 (1989).
9. S. Weinberg, *The Quantum Theory of Fields*, vol. 1, Cambridge Univ. Press, Cambridge (1995).
10. T.D. Newton and E. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
11. L.P. Horwitz, C. Piron and F. Reuse, Helv. Phys. Acta **48**, 546 (1975); R. Arshansky and L.P. Horwitz, J. Phys. A: Math. Gen. **15**, L659-662 (1982).
12. A. Palacios, T.N. Rescigno, and C.W. McCurdy, Phys. Rev. Lett. **103**, 253001-1-253001-4 (2009).
13. M.C. Land and L.P. Horwitz, J. Phys. A: Math and Gen **28**, 3289 (1995).
14. See, for example, H. Boerner, *Representations of Groups*, North Holland, Amsterdam (1963).
15. P.A. M. Dirac, *The Principles of Quantum Mechanics*, Clarendon Press, Oxford (1948).
16. J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics*, McGraw Hill, New York (1964).
17. D. Saad, L.P. Horwitz and R.I. Arshansky, Found. Phys. **19**, 1125-1149 (1989). See also, I Aharonovich and L.P. Horwitz, Jour. Math. Phys. **47**, 122902 (2006); **51**, 052903-1-052903-27 (2010); **52**, 082901-1-082901-11 (2011); **53**, 032902-1-032902-29 (2012); Eur. Phys. Lett. **97**, 60001-p1-60001-p3 (2012). See also, J.D. Jackson, *Classical Electrodynamics*, 2nd edition, Wiley, New York (1975).
18. J. Schwinger, Phys. Rev. **82**, 664 (1949).
19. A.F. Bennett, J. Phys. A: Math. Theor. **45**, 285302 (2012).
20. See V.E. Koperin, N.M. Bogliubov and A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge Monographs on Mathematical Physics, Cambridge (1993).
21. For example, A.L. Fetter and J.D. Walecka, *Quantum Theory of Many Particle Systems*, McGraw-Hill, New York (1971).
22. H. Fröhlich, Phys. Rev. **79**, 845 (1950); L.N. Cooper, Phys. Rev. **118**, 1189 (1956).
23. B.D. Josephson, Phys. Lett. **1**, 251 (1962).
24. F. Lindner *et al* Phys. Rev. Lett. **95**, 040401 (2005).
25. J. Schwinger, Phys. Rev. **74**, 1439 (1948).
26. S. Tomanaga, Prog. Theor. Phys. **1**, 27 (1946).
27. J.M. Jauch and F. Rohrlich. *The Theory of Photons and Electrons*. 2nd edition, Springer-Verlag, New York (1976).

28. L. Horwitz, *Phy. Lett. A* **355**, 1-6 (2006).